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THE WORD PROBLEM IN A CLASS OF NON-HAKEN 3-MANIFOLDS

ANDREW SKINNER

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1. INTRODUCTION

THE Word Problem for a given finite presentation $\langle X \mid R \rangle$ of a group G is said to be solvable if it is possible to decide whether or not a word ω in the generators X represents the trivial element in G [31]. If the Word Problem is solvable for a particular finite presentation of G, then it is solvable for all other finite presentations and we say that the Word Problem in G is solvable.

Many groups (e.g. free groups and 1-relator groups) are known to have solvable Word Problems, while on the other hand there are examples of finitely presented groups in which it is known that the Word Problem is not solvable [31]. From a topological point of view, it is natural to consider the Word Problem for the fundamental groups of topological spaces, and in manifolds in particular. We say that a manifold, M, whose fundamental group is finitely presented, has a solvable Word Problem if the Word Problem in its fundamental group is solvable.

There is however a more geometric way of formulating the Word Problem which is more suitable to the topologial situation. For simplicity we will restrict ourselves in everything which follows to the category of smooth manifolds and maps. In the three dimensional case, with which we will be mainly concerned, the smooth and piecewise linear categories are interchangeable [23]. The Word Problem in a smooth manifold, M, is said to be solvable if, given any immersion $f: S^1 \to M$, it is possible to decide whether or not there exists an immersion $g: D^2 \to M$ (a *singular disc*) such that $g \mid \partial D^2 = f$. This is essentially saying that it is possible to decide whether or not an immersed closed curve in M is contractible. If we observe that, with a suitable choice of basepoint, any loop in M represents an element of $\pi_1(M)$ and that this element is trivial if and only if the loop is contractible, it is clear that this geometric form of the Word Problem is equivalent to the original algebraic formulation in terms of a given finite presentation of $\pi_1(M)$.

Note that a solution of the Word Problem is usually presented in the form of an algorithm which will decide in a finite number of steps whether or not any given loop in M is contractible.

It is of interest to topologists to determine which classes of manifolds have solvable Word Problems [22]. It is known that, while all compact two-dimensional manifolds (surfaces) have solvable Word Problems [41], there are examples, in dimension four and above, of manifolds which are known not to have solvable Word Problems [22]. In dimension three, however, the situation is still unresolved. While it is not known that all three-dimensional manifolds have solvable Word Problems, there are no known examples in which it is known not to be solvable. Indeed, the solvability of the three dimensional word problem would be a consequence of Thurston's Geometrisation Conjecture [43].

However solutions for the Word Problem, not depending upon any geometric decomposition, do exist for large classes of 3-manifolds. In particular Waldhausen gave an algorithm to solve the Word Problem in the class of Haken manifolds [46] which also applies to all 3-manifolds which are finitely covered by a Haken manifold, and so in particular to the class of all non-Haken, Seifert fibred spaces with infinite fundamental group [39].

In the following we extend the class of 3-manifolds known to have solvable Word Problems to those non-Haken manifolds which contain a singular incompressible surface satisfying the so-called 1-line and 1-point intersection properties. An example of a manifold in this class is a non-Haken, hyperbolic manifold containing an immersed totally geodesic surface. In this case the Word Problem is already known to be solvable, since the manifold has a negatively curved metric. (Also Long [28] shows that in such a situation the 3-manifold has a finite cover which is Haken, so that Waldhausen's solution can be applied.) It will be shown that other examples which satisfy these conditions can be generated by certain surgeries on link complements which admit a cubing of non-positive curvature [3].

2. PRELIMINARIES

The book of Hempel [23] and the articles of Hass and Scott [21] and Hass, Scott and Rubinstein [20] provide good references for much of the following material. We will assume, unless stated otherwise, that all manifolds and maps are smooth.

A compact 3-manifold, M, is said to be *closed* if $\partial M = \emptyset$ and *irreducible* if any embedded 2-sphere in M bounds a ball in M. A compact, irreducible 3-manifold, M, is P^2-*irreducible* if it contains no 2-sided embedding of the projective plane, P^2. A compact, 2-sided surface, $F \neq S^2$, P^2, properly embedded in a compact, P^2-irreducible 3-manifold M (so that $\partial F = F \cap \partial M$), is said to be *incompressible* if there is no embedded disc, D, in M with $D \cap F = \partial D$, unless ∂D bounds a disc in F.

A compact, P^2-irreducible 3-manifold is said to be Haken if it contains an embedded incompressible surface, F. Since Waldhausen's solution of the Word Problem applies to Haken manifolds [46], we will only consider the case of 3-manifolds which are *non-Haken*, in other words those compact, P^2-irreducible 3-manifolds which contain no incompressible surface. It is well known that any compact, P^2-irreducible 3-manifold, $M \neq B^3$, with $\partial M \neq \emptyset$, is Haken [23]. It follows that all compact, P^2-irreducible, non-Haken 3-manifolds are closed. Furthermore it is known that a closed, P^2-irreducible, non-Haken 3-manifold is necessarily orientable [Heil].

Let M be a closed P^2-irreducible, non-Haken (and hence orientable) 3-manifold and $F \neq S^2, P^2$ a closed surface. If $f\colon F \to M$ is an immersion, the singular set of f is defined to be the set

$$S(f) = \{x \in F \mid f(x) = f(y) \text{ for some } y \in F, y \neq x\}.$$

If f is a general position immersion and F is orientable then $S(f)$ consists of finitely many immersed closed curves intersecting transversely and identified in pairs by f. We will refer to the images of these curves under f as *double curves*. The double points of these curves on F correspond to the triple points of the image $f(F)$ in M.

A map, $f\colon F \to M$, is π_1-*injective* if the induced map $f_\#\colon \pi_1(F) \to \pi_1(M)$ is an injection. We will also refer to $f\colon F \to M$ as a *singular incompressible surface*. Since M is non-Haken it is not possible that $S(f) = \emptyset$. Given a Riemannian metric on M, we say that $f\colon F \to M$ is *least-area* if it minimizes area in its homotopy class. A result of minimal surface theory says that there is a least-area representative in the homotopy class of a π_1-injective map,

$f: F \to M$, and that this least-area map, $g: F \to M$, is an immersion [36]. The work of Freedman, Hass and Scott [13] shows that the pre-image, $p^{-1}(g(F))$, of $g(F)$ under the universal covering projection, $p: \tilde{M} \to M$, consists of a family, $\Pi = \{P_i\}$, of embedded planes in $\tilde{M}$, any two of which intersect in a collection of (embedded) lines. Each plane, $P \in \Pi$, is the image, $g(\mathbf{R}^2)$, of some lift, $\tilde{g}: \mathbf{R} \cong \tilde{F} \to \tilde{M}$, of g and hence is a homeomorphic copy of the universal cover, $\tilde{F} \cong \mathbf{R}^2$, of F. In fact Π is just the orbit of $\tilde{g}(\mathbf{R}^2)$ under the action of the group of covering transformations of $\tilde{M}$. The pre-image of $S(f)$ on each $P \in \Pi$ is a family of embedded lines

$$\Lambda_P = \{l \mid l \text{ is a component of } Q \cap P \text{ where } Q \in \Pi,\ Q \neq P\}, \quad \text{on } P.$$

Note that there is a corresponding commuting diagram,

$$\begin{array}{ccc} \tilde{F} \cong \mathbf{R}^2 & \xrightarrow[\cong]{\tilde{g}} & \tilde{g}(\tilde{F}) = P \subset \tilde{M} \cong \mathbf{R}^3 \\ \downarrow & & \downarrow p \\ F & \xrightarrow{g} & g(F) \subset M \end{array}$$

Although the least-area map, $g: F \to M$, may not be in general position, it is possible to perturb g slightly to a general position immersion, $g': F \to M$, so that $\Pi' = p^{-1}(g'(F))$ is again a family of embedded planes in $\tilde{M}$ any two of which intersect transversely in a collection of lines [20]. Although g' is no longer, in general, a least-area map, the planes, Π', will be in general position. A general position family of embedded planes, Π, in $\tilde{M}$, any two of which intersect transversely in a (possible empty) collection of lines, is said to be a *simple* family of planes [20].

The preceding discussion can be summarized in the following result:

THEOREM ([21], [20]). *Let M be a closed, P^2-irreducible, non-Haken 3-manifold, $F \neq S^2$, P^2 a closed surface and $f: F \to M$ a π_1-injective map. Then there exists a map, $g: F \to M$, homotopic to f, such that the full pre-image, $p^{-1}(g(F))$, of $g(F)$ in $\tilde{M}$ is a simple family of planes.*

Hass, Rubinstein and Scott [20] use this result to show that $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$. Furthermore they show that closure of each component of $M - g(F)$ is a π_1-injective handlebody in M.

Using the terminology of Hass and Scott [21], if $f: F \to M$ is a map such that the pre-image, $p^{-1}(f(F))$, of $f(F)$ in $\tilde{M}$ is a simple family of planes, Π, then we say that Π satisfies,

1. the 1-*line intersection property* if two planes from Π are either disjoint or intersect in a single line,
2. the 4-*plane intersection property* if given any four distinct planes from Π at least two do not intersect,
3. the 1-*point intersection property* if any three planes from Π have at most one point in common,
4. the *triple-point property* if, given three planes from Π which intersect pairwise, any homotopy of f to a map, g, such that $p^{-1}(g(F))$ is a simple family of planes has the property that the image of the three planes under the induced homotopy of Π have a point in common.

Note that if $g: F \to M$ lifts to a simple family of planes in $\tilde{M}$ satisfying the 1-line and 1-point intersection properties, then any two lines from the system, Λ_P, on a plane $P \in \Pi$ can intersect in at most one point. In Section 4 we will be concerned mainly with this case and will prove the following, which is the main result of the paper:

THEOREM 4.11. *Let M be a closed, P^2-irreducible, non-Haken 3-manifold which admits a π_1-injective immersion, $f: F \to M$, of a closed surface $F \neq S^2, P^2$, such that the full pre-image, $p^{-1}(f(F))$, of $f(F)$ in $\tilde{M}$ is a simple family of planes satisfying the 1-line and 1-point intersection properties. Then the Word Problem in M is solvable.*

We will however also consider two special cases of this result which give a more efficient solution of the Word Problem. In particular, in the list of assumptions of Theorem 4.11, we will replace the 1-point property first by the 4-plane property and then by the 4-plane and triple-point properties together. Although it is not obvious, the 1-line and 4-plane properties, taken together, imply the 1-point property, as Hass and Scott [21] prove in the following:

THEOREM 3.1 (of [21]). *Let M be a closed, P^2-irreducible, non-Haken 3-manifold, which admits a π_1-injective immersion, $f: F \to M$, of a closed surface, $F \neq S^2, P^2$, such that the full pre-image, $p^{-1}(f(F))$, of $f(F)$ in M is a simple family of planes satisfying the 1-line and 4-plane intersection properties. Then f is homotopic to an immersion, g, such that $p^{-1}(g(F))$ is a simple family of planes in $\tilde{M}$ satisfying the 1-line, 4-plane and 1-point intersection properties.*

It is an immediate consequence of this result that the solution of the Word Problem given in Theorem 4.11 applies to the 1-line and 4-plane case. However if we adapt the proof of Theorem 4.11 by explicitly using the 4-plane property, we can achieve an algorithm which will in general terminate in fewer steps.

Note that if the map, f, in Theorem 3.1 of Hass and Scott [21], is assumed to satisfy the triple-point property in addition to the 4-plane and 1-line properties then, from the definition in 4 above, the map, g, will also satisfy the triple-point property. We show in Lemmas 5.3 and 5.4 that Theorem 4.11 can be adapted in the presence of the triple-point property to give an even faster algorithm.

In Section 7 we mention some non-trivial examples of 3-manifolds for which the solution to the Word Problem given in Theorem 4.11 is appropriate.

3. A TECHNICAL RESULT

As noted in the previous section, Hass, Rubinstein and Scott [20] show that the closure of each component of the image $f(F)$ of a π_1-injective surface $f: \to M$ is a π_1-injective handlebody $X \subset M$. Each such handlebody is bounded by $f(F)$ and has a graph J inscribed on its surface by the double curves of f. Each vertex of J will have degree at most three.

In order to construct an algorithm to shrink a closed loop α in M we examine the behaviour of pieces of α in these handlebody components of M. In particular since each handlebody X is π_1-injective and embedded in M a homotopy of α can be performed in M if and only if it can be performed in X.

Let M be a Haken 3-manifold with (possibly empty) boundary and J a finite graph in ∂M in which each vertex has degree at most three. The following results are based on the work of Waldhausen [46]. An incompressible, boundary incompressible surface F in M is said to be *good* with respect to J if the following properties hold:

1. ∂F is in general position with respect to J.
2. Let D be a disc in M with $D \cap (F \cup \partial M) = \partial D$ and such that $D \cap F$ is an arc α in ∂D with $\alpha \cap \partial F = \partial \alpha$. Then if $D \cap J$ consists of at most one point there exists a disc D' in F such that $\partial D' \subset \alpha \cup \partial M$ and such that $D' \cap J$ consists of not more points than $D \cap J$.
3. F has maximal Euler characteristic among all surfaces with the second property.

LEMMA 3.1. *If M is a Haken 3-manifold not homeomorphic to a finite collection of balls and J a finite graph of degree at most three in ∂M (which may be empty) then there exists a good surface F in M. Furthermore F can be constructed.*

A *singular disc* in M is a map $f: D^2 \to M$. Waldhausen [46] proves the following two results about the behaviour of singular discs in the presence of a good surface:

LEMMA 3.2. *If F is a good surface in M and $f: D \to M$ a singular disc such that $f^{-1}(F) = \partial D$ then there exists a singular disc $g: D \to M$ such that $g | \partial D = f | \partial D$ and $g(D) \subset F$.*

LEMMA 3.3. *Let F be a good surface with respect to J in M and let $f: D \to M$ be a singular disc such that $f^{-1}(F)$ is an arc α in ∂D with $f^{-1}(\partial M) = \partial D - int(\alpha)$ and such that $f^{-1}(J)$ is at most one point disjoint from $\partial\alpha$. Then there exists a singular disc $g: D \to M$ such that $g | \alpha = f/\alpha$, $g(D) \subset F$, $g(\partial D - \alpha) \subset \partial M$ and such that $g^{-1}(J)$ consists of no more points than $f^{-1}(J)$.*

Let X be a handlebody and J a finite graph of degree three on ∂X. Let $v(J)$ denote the set of vertices of J. Assume that J is *non-degenerate* in the sense that if $D \subset X$ is a properly embedded disc in X such that $|\partial D \cap J| \leq 2$ then there is a disc $D' \subset \partial X$ with $\partial D' = \partial D$ such that $D' \cap J$ is either empty or a subarc of the interior of an edge in J.

Let $F = \bigcup D_i$ be a collection of properly embedded, mutually disjoint discs in X such that $X - \mathcal{N}(F)$ is a collection of balls. Clearly F is incompressible, boundary incompressible and can be chosen to be good with respect to J. Furthermore we can choose F so that it has a minimal number of components.

Let $x, y \in \partial X - (J \cup \partial F)$ and for $i = 1$ or 2 let $\beta_i: I \to \partial X$ be an arc in general position with respect to $J \cup \partial F$ such that,

1. $\beta_i(0) = x$ and $\beta_i(1) = y$,
2. $|\beta_i^{-1}(J)| \leq 1$.

Suppose that β_1 and β_2 are homotopic within X but are not homotopic within $\partial X - v(J)$. Let $f: D \to X$ be a singular disc in general position with respect to F such that $f | \partial D = \beta_1 \cup \beta_2$ and $f^{-1}(\partial X) = \partial D$.

Since F is incompressible we may assume that no component of $f^{-1}(F)$ is a loop and hence that $f^{-1}(F)$ is a minimal collection of mutually disjoint properly embedded arcs on D. Choose an innermost component α of $f^{-1}(F)$ and let $D' \subset D$ denote the disc split off by α with the property that $D' \cap f^{-1}(F) = \alpha$. Clearly $f | \partial D' - \alpha$ is a subarc of $\beta_1 \cup \beta_2$ and hence $f^{-1}(J) \cap D' \subset \partial D$ consists of at most two points. However, if $f^{-1}(F)$ is not empty we may always choose α so that $f^{-1}(J) \cap \partial D'$ consists of at most one point. By Lemma 3.3 there exists a singular disc $g: D' \to X$ such that $g | \alpha = f | \alpha$, $g(D) \subset F$, $g(\partial D' - \alpha) \subset \partial X$ and such that $g^{-1}(J)$ consists of no more points than $f^{-1}(J) \cap D'$.

We show that there is a homotopy between $g: D' \to X$ and $f|D': D' \to X$ which induces a homotopy, in $\partial X - v(J)$, between the arcs $g|\partial D' - \alpha$ and $f|\partial D' - \alpha$. To see this, let $\mathcal{N}(F)$ be a regular neighbourhood of F in X, and let Y denote the closure of the component of $X - \mathcal{N}(F)$ containing the image of $f|D'$. Since Y is a ball and F a union of properly embedded discs in X, $\partial X \cap \partial Y$ is a planar surface A. Each component of $A - J$ is simply connected since otherwise it would be possible to find a disc, D, properly embedded in Y, with $\partial D \subset A - J$ and such that there is no disc, D', in $\partial X - J$ with $\partial D' = \partial D$. Clearly this would contradict the assumption of non-degeneracy of J. As a result we can find an embedded arc, $\theta: I \to A$, homotopic to $f|\partial D' - \alpha$(relative endpoints) such that $\theta(I) \cap J = (f|\partial D')(I) \cap J$. Since Y is a ball and $\theta \cup g|\partial D' - \alpha$ an embedding, there is a properly embedded disc $h: D \to Y$ with $h|\partial D = \theta \cup g|\partial D' - \alpha$. But by assumption there must then be a disc $D'' \subset \partial X$ with $\partial D'' = h(\partial D)$ such that $D'' \cap J$ is either empty or a subarc of the interior of an edge in J. This gives the required homotopy.

Let $f_1: D \to X$ denote the singular disc obtained by replacing $f|D'$ by g and then moving the resulting singular disc into general position with respect to F, in such a way that $f_1^{-1}(F)$ contains a minimal number of components. We can repeat this procedure (at most a finite number of times) until we obtain a singular disc $f_n: D \to X$ such that $f_n^{-1}(F) = \emptyset$, $f_n^{-1}(\partial X) = \partial D$ and $f_n^{-1}(J)$ consists of at most two points. Let Y denote the closure of the component of $X - \mathcal{N}(F)$ containing the image of f_n. As before, Y is a ball and $A = \partial X \cap \partial Y$ a planar surface with each component of $A - J$ simply connected. Let $\gamma = f_n|\partial D$. It follows from the construction of f_n that γ is homotopic to $\beta_1 \cup \beta_2$ in $\partial X - v(J)$. Since $\gamma^{-1}(J)$ consists of at most two points, it is clear that there is a properly embedded disc, $h: D \to Y$, with $h(D) \cap J = \gamma(\partial D)J$, such that $\gamma' = h|\partial D$ is a loop in A, homotopic to γ in $\partial Y - v(J)$. But by assumption, there must then be a disc $D'' \subset \partial X$, with $\partial D'' = h(\partial D)$, such that $D' \cap J$ is either empty or a subarc of the interior of an edge in J. It follows that γ is homotopic to a point in $\partial X - v(J)$ and hence that β_1 and β_2 are homotopic (relative endpoints) in $\partial X - v(J)$.

Let $\alpha: I \to X$ be a properly embedded arc with endpoints, x and y, in $\partial X - J$. Then the above argument shows that, up to homotopy within $\partial X - v(J)$, there exists at most one arc $\beta: I \to \partial X$, in general position with respect to J such that $\beta^{-1}(J)$ is at most one point. Furthermore, Waldhausen's algorithm for the Word Poblem in X [46] shows that it is possible to decide whether or not β exists, and if so, to construct it.

The result of this section can be summarized in the following:

THEOREM 3.4. *Let $\alpha: I \to X$ be a properly embedded arc with endpoints, x and y, in $\partial X - J$. Then up to homotopy within $\partial X - v(J)$, there exists at most one arc $\beta: I \to \partial X$, in general position with respect to J such that $\beta^{-1}(J)$ is at most one point.*

4. THE WORD PROBLEM ALGORITHM

Let M^3 be a closed, P^2-irreducible, non-Haken 3-manifold and $f: F \to M$ a general position immersion of a closed surface, $F^2 \neq S^2, P^2$, into M such that the full pre-image of $f(F)$ in the universal cover, $\tilde{M}$, of M is a simple family of planes $\Pi = \{P_i\}$ satisfying the 1-line and 1-point intersection properties. Let K denote the 2-complex, $\cup_i P_i$, obtained by taking the union of planes in Π and, for $P \in \Pi$, let Λ_P denote the family of lines, $\{P \cap Q | Q \in \Pi, Q \neq P\}$, on P. Let L_P denote the 1-complex on P obtained by taking the union of the lines Λ_P and let $L = \bigcup_{P \in \Pi} L_P$.

If N is a smooth manifold, we will refer to a map $\alpha: I \to N$ as an *arc* in N. The *endpoints* of α are the points $\alpha(0)$ and $\alpha(1)$ in N. If $\alpha(0) = \alpha(1)$ we can consider α as a map $\alpha: S^1 \to N$

and refer to it as a *loop* in N. To simplify the notation used we will usually use the same symbol α to refer to an arc and its image in N. The intended interpretation should be clear from the context and causes no loss of generality if α is an embedding.

Let α be an arc in M which is in general position with respect to $f(F)$. In particular $\alpha \cap S(f) = \emptyset$ and $n(\alpha) = |\operatorname{int}(\alpha) \cap f(F)|$ is finite. A sub-arc of α is said to be a *segment* of α if it has endpoints on $f(F)$ and interior disjoint from $f(F)$. Similarly if α is an arc in $\tilde{M}$ which is in general position with respect to Π then $\alpha \cap L = \emptyset$ and $n(\alpha) = |\operatorname{int}(\alpha) \cap K|$ is finite. A sub-arc of α is said to be a *segment* of α if it has endpoints on K and interior disjoint from K. If α has endpoints x and y on K then its *initial segment* is the segment of α which includes x. We say that an arc α in $\tilde{M}$ which is in general position with respect to Π is *outermost* if both of its endpoints lie on some plane $P \in \Pi$ and if it has no proper sub-arc with both endpoints on a single plane of Π.

The following two results are straightforward:

LEMMA 4.1. *Let $P \in \Pi$ and $x, y \in P - L_P$. A line, $l = Q \cap P \in \Lambda_P$, separates x and y on P if and only if Q separates x and y in M.*

Proof. This is an immediate consequence of the 1-line property. Q.E.D.

LEMMA 4.2. *Let α be an outermost arc with $n(\alpha) = k$, whose endpoints, x and y, lie on a plane $P \in \Pi$. Then $\operatorname{int}(\alpha)$ intersects exactly k planes, $P_1, \ldots, P_k \in \Pi$, each in a single point. Furthermore x and y are separated on P by the k distinct lines, $l_1 = P \cap P_1, \ldots, l_k = P \cap P_k$.*

Proof. If $Q \neq P$ is a plane from Π with $|\alpha \cap Q| > 1$, then α must have a proper sub-arc with both endpoints on Q, contradicting the assumption that α is outermost. It follows since $n(\alpha) = k$, that α intersects exactly k distinct planes, $P_1, \ldots, P_k \in \Pi$, each in a single point. Hence x and y are separated in M by the planes $P_1, \ldots, P_k$ and no others. By Lemma 4.1, x and y are separated on P by the lines, $l_1 = P \cap P_1, \ldots, l_k = P \cap P_k$. These lines are distinct since Π is a general position family of planes. Q.E.D.

If α is an embedded arc on $P \in \Pi$ with endpoints $x, y \in P - L_P$ and which is in general position with respect to Λ_P then we let $n(\alpha) = |\alpha \cap L_P|$.

LEMMA 4.3. *Let $P \in \Pi$ and let $x, y \in P - L_P$ be separated on P by the lines, $l_1, \ldots, l_k$, and on other lines $l \in \Lambda_P$. Then there exists an embedded arc, α, from x to y on P, which is in general position with respect to Λ_P and such that $n(\alpha) = k$. In particular, α meets each of the lines, $l_1, \ldots, l_k$, in exactly one point and meets no other lines of Λ_P.*

Proof. Let α be an embedded arc on P, from x to y, which is in general position with respect to Λ_P, with the property that $n(\alpha)$ is minimal among all such arcs. Suppose that

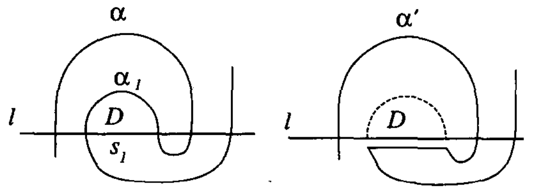

Fig. 1. Pulling α across l to reduce $\#(\alpha \cap l)$.

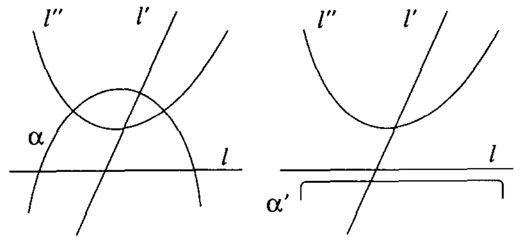

Fig. 2. $n(\alpha') < n(\alpha)$.

$\#(\alpha \cap l) = n \geq 2$ for some $l \in \Lambda_P$. Let $x_1, \ldots, x_n$ denote the points of $\alpha \cap l$ in order along α from x to y. Since α is embedded, for some $1 \leq i < n$ the segment, s_i, of l between x_i and x_{i+1} contains none of the points, $x_1, \ldots, x_n$, in its interior. Therefore, the interior of the 2-gon region, D, bounded by s_i and the arc, α_i, of α between x_i and x_{i+1} is disjoint from both α and l. Using D we pull α across l as shown in Fig. 2, to obtain a new arc, α', which intersects l in fewer points than α.

Since Π satisfies the 1-point property, any other line, $l' \in \Lambda_P$, which meets D can intersect s_i in at most one point. It follows that α' will intersect the other lines of Λ_P in at most as many points as does α (Fig. 2). Hence $n(\alpha') < n(\alpha)$, contradicting the assumption that $n(\alpha)$ was minimal.

It follows that $\#(\alpha \cap l) \leq 1$ for each line $l \in \Lambda_P$. *But* $\#(\alpha \cap l) = 1$ if and only if l separates x from y, and hence $n(\alpha) = k$. Q.E.D.

4.1. Compressions of outermost arcs

Let α be an outermost arc with $n(\alpha) = k$ whose endpoints x and y lie on a plane, $P \in \Pi$, and let α' be an embedded arc from x to y on P which is in general position with respect to Λ_P and such that $n(\alpha') = k$. The existence of α' follows from Lemma 4.2 and Lemma 4.3. Denote by $P_1, \ldots, P_k$, the k distinct planes from Π, each of which $\mathrm{int}(\alpha)$ intersects in a single point. It follows that α' intersects the k distinct lines, $l_1 = P \cap P_1, \ldots, l_k = P \cap P_k$ from Λ_P, again each in a single point.

A singular disc, $g\colon D \to \tilde{M}$ in general position with respect to Π with $g|\partial D = \alpha \cup \alpha'$ is said to *span* α (Fig. 3) if,

1. $\mathrm{int}(D) \cap g^{-1}(Q) = \emptyset$, unless Q is one of the planes $P_1, \ldots, P_k$ and,
2. $g^{-1}(P)$ is an arc s in ∂D with $g|s = \alpha'$.

If $T \subset D$ is the closure of a component of $D - g^{-1}(K)$ then it is said to be an *upper triangle* of g if,

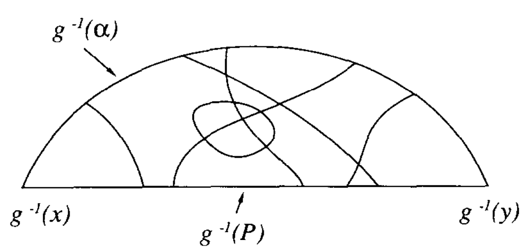

Fig. 3. A disc which spans α.

1. T is 3-gon region of the graph $\partial D \cup g^{-1}(K)$ on D,
2. $s' = T \cap \partial D - \mathrm{int}(s)$ is a non-empty arc,
3. $g|s'$ is a non-initial segment of α.

An outermost arc α is *simple* if $n(\alpha) = 0$. A disc $g\colon D \to \tilde{M}$ which spans a simple outermost arc is said to be a *simple disc.*

If a disc g has an upper triangle T then it may be modified by an *elementary compression* to give a new disc $g' = g|_{D'}$ where $D' = D - \mathcal{N}(T)$ for some open neighborhood $\mathcal{N}(T)$ of T as shown in Fig. 5. The loop $g'|\partial D'$ is homotopic to $\alpha \cup \alpha'$ and can be written as $\beta \cup \beta'$ where β is outermost with endpoints on P and β' is a sub-arc of α' with $n(\beta') = n(\beta) \leq n(\alpha)$. Clearly g' spans β.

A *triangular compression* of $g\colon D \to \tilde{M}$ is a sequence of discs

$$g_0 = g\colon D_i = D \to \tilde{M}, g_1\colon D_1 \to \tilde{M} \cdots g_n\colon D_n \to \tilde{M}$$

such that,

1. for each $i = 1, \ldots, n-1$, the disc g_{i+1}, is obtained from g_i by an elementary compression.
2. g_n spans a simple arc α_n whose initial endpoint is x.

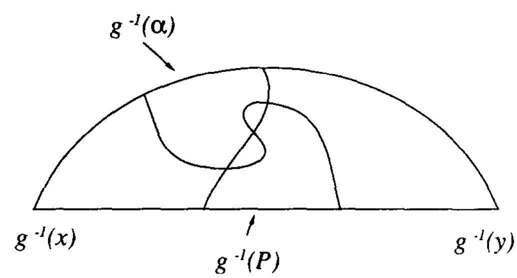

Fig. 4. A disc D with no upper triangle.

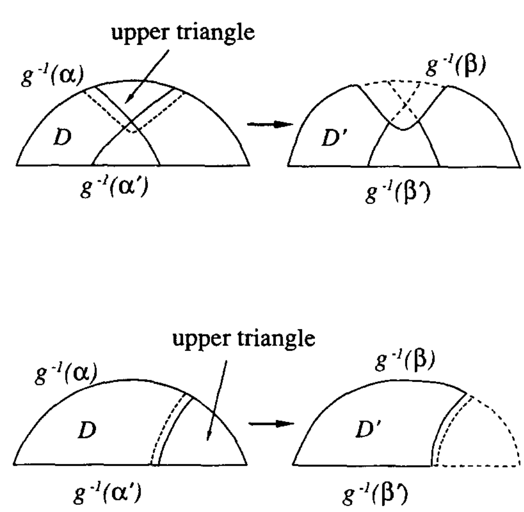

Fig. 5. Modifying g by an elementary compression.

It follows that for each $i = 0, \ldots, n-1$ there is an upper triangle $T_i \subset D_i$ of g_i such that $D_{i+1} = D_i - \mathcal{N}(T_i)$ and that $g_{i+1} = g_i | D_{i+1}$.

We let $l(g) = n + 1$ denote the *length* of the triangular compression, $g_0 = D, g_1, \ldots, g_n$.

LEMMA 4.4. *If Π satisfies the 1-line and 1-point intersection properties then any outermost arc α with $n(\alpha) = k$ is spanned by a disc $g: D \to \tilde{M}$ which admits a triangular compression of length $l(g) \leq 2^k$.*

Proof. We argue by induction on $n(\alpha)$.

If $n(\alpha) = 0$ the result is clear since α will certainly be spanned by a disc $g: D \to \tilde{M}$. But g trivially admits a triangular compression of length $l(g) = 1$ since α is certainly simple.

Suppose that $n(\alpha) = 1$. Let α have endpoints x and y on $P \in \Pi$ and let $\mathrm{int}(\alpha)$ meet the plane $P_1 \in \Pi$. Using Lemma 4.3, choose an arc α' from x to y on P with $n(\alpha') = 1$ so that the loop $\alpha \cup \alpha'$ meets P_1 in two points $x_1 = \alpha' \cap P_1$ and $y_1 = \alpha' \cap P_1$. We can choose an embedded arc β_1 from x_1 to y_1 on P_1 with $n(\beta_1) = 0$ and a disc $g: D \to \tilde{M}$ spanning α such that $s = g^{-1}(P_1)$ is a properly embedded arc on D with $g|s = \beta_1$. Observe (Fig. 6) that q has an upper triangle T and that an elementary compression of T gives a disc $g_1: D_1 \to \tilde{M}$ spanning a simple arc whose initial endpoint is x. Thus $q = g_0, g_1$ is a triangular compression of g with $l(g) = 2$.

Suppose now that the claim is true for all outermost arcs β with $n(\beta) < k$.

Let α be an outermost arc with $n(\alpha) = k$ whose endpoints x and y lie on a plane $P \in \Pi$ and let α' be an embedded arc from x to y on P, which is in general position with respect to Λ_P and such that $n(\alpha') = k$. Denote by $P_1, \ldots, P_k$ the k distinct planes from Π which $\mathrm{int}(\alpha)$ meets.

Let γ be the loop obtained by a small general position perturbation of $\alpha \cup \alpha'$ off P. Without loss of generality let P_1 be the first plane (after x) which α' meets and let $x_1 = \alpha \cap P_1$ and y_1 be the two points of $\gamma \cap P_1$.

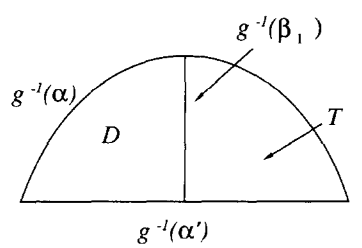

Fig. 6.

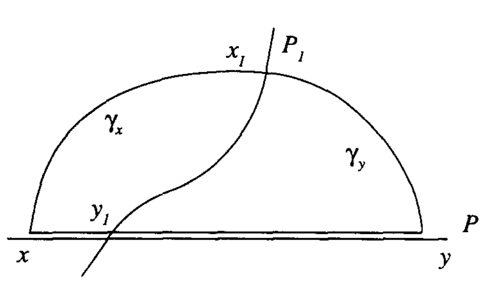

Fig. 7.

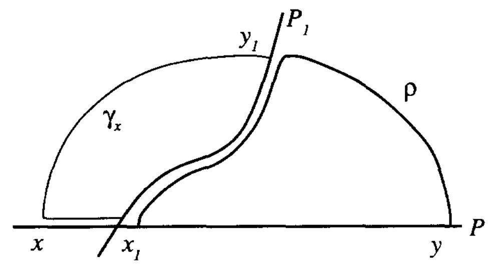

Fig. 8.

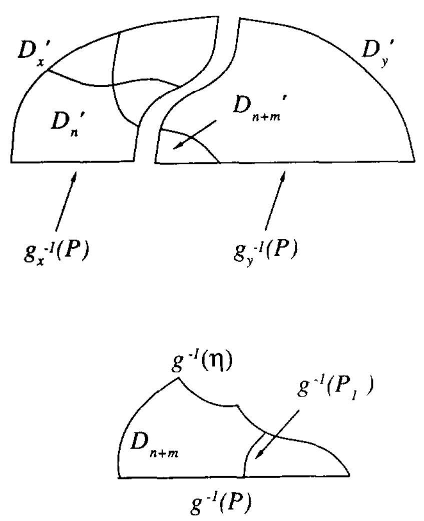

Fig. 9.

Denote by γ_x and γ_y the two arcs of γ from x_1 to y_1 as shown in Fig. 7 with γ_x containing x. Clearly γ_x is an outermost arc with endpoints x_1 and y_1 on P_1 with $n(\gamma_x) < k$. It follows that there is a disc $g_x\colon D_x \to \tilde{M}$ spanning γ_x which admits a triangular compression,

$$g_x = g_0'\colon D_0' \to \tilde{M}, \dots, g_n'\colon D_n' \to \tilde{M}$$

with $l(g_x) = n + 1 \le 2^{n(\gamma_x)} \le 2^{k-1}$.

Observe that $s = g_x^{-1}(P_1)$ is an arc in ∂D_x and that $\mu = g_x|s$ is an embedded arc from x_1 to y_1 on P_1 with $n(\mu) = n(\gamma_x) < k$. Let ρ denote the arc from x_1 to y obtained by perturbing a subarc of $\mu \cup \gamma_y$ slightly as shown in Fig. 8. It is clear that ρ is outermost and that $n(\rho) = n(\mu) + k - n(\gamma_x) - 1 = k - 1$. Hence there is a disc $g_y\colon D_y \to \tilde{M}$ spanning ρ which admits a triangular compression

$$g_y = g_{n+1}'\colon D_{n+1}' \to \tilde{M}, \dots, g_{n+m+1}'\colon D_{n+m+1}' \to \tilde{M}$$

with $m + 1 \le 2^{k-1}$.

For $i = 0, 1, \dots, n$ let $D_i = D_i' \cup D_{n+1}'$ and let $g = g_x \cup g_y\colon D_0 \to \tilde{M}$ where it is understood that g_x and g_y have been perturbed slightly so that g spans α. For $i = n + 1, \dots, n + m$ let $D_i = D_{i+1}' \cup D_n'$ and for $i = 1, \dots, n + m$ let $g_i = g|D_i$. Clearly $g_0, g_1, \dots, g_n, g_{n+1}, \dots, g_{n+m}$ is a sequence of discs such that for each $i = 0, 1, \dots, n + m - 1$ the disc g_{i+1} is obtained from g_i by an elementary compression.

Observe that $g_{n+m} = g|_{D_{n+m}}$ is a disc spanning an outermost arc η with $n(\eta) = 1$ and endpoints on P (Fig. 9). Furthermore the initial endpoint of η is x. Clearly $g_{m+n}, g_{m+n+1} = g'_n$ where $g'_n = g|_{D'_n}$ is a triangular compression of $g|_{D_{m+n}}$ since $g|_{D'_n}$ is a simple arc whose initial endpoint is x.

It follows that $g_0, \ldots, g_{n+m+1}$ is a triangular compression of $g: D_0 \to \tilde{M}$ with length $l(g) = n + m + 2 \leq 2^{k-1} + 2^{k-1} = 2^k$. Q.E.D.

4.2. *The algorithm*

LEMMA 4.5. *Let γ be a loop in $\tilde{M}$ which is in general position with respect to K and such that $n(\gamma) = |\gamma \cap K| = 2k$. Then there exists a sub-arc α of γ which is outermost and such that $n(\alpha) \leq k - 1$.*

Proof. If α is an arc of γ with endpoints on a plane $P \in \Pi$ and which is not outermost then there must exist a subarc α' of α with $n(\alpha') < n(\alpha)$ and with both endpoints on a single plane of Π. If α' is not outermost then there must exist a subarc α'' of α' with $n(\alpha'') < n(\alpha')$ and with both endpoints on a single plane of Π. We may continue with this procedure until we find an arc which is outermost.

To see that $n(\alpha) \leq k - 1$, note that γ can intersect at most k planes from Π. It follows that $\mathrm{int}(\alpha)$ can intersect at most $k - 1$ planes each in a single point. Q.E.D.

Let X be the closure of a component of $M - f(F)$ and $\alpha: I \to X$ a properly embedded arc with endpoints $x, y \in \partial X - f(S(f))$. An *elementary compression (resp. a simple compression)* of α is a disc $g: D \to M$ such that,

1. $\alpha = g|s$ for some arc $s \subset \partial D$,
2. $\alpha' = g|\partial D - \mathrm{int}(s)$ in an arc from x to y on $f(F)$ with $|\alpha'^{-1}(\ (S(f)))| = 1$ (resp. $|\alpha'^{-1}(f(S(f)))| = 1$) which lifts to an embedding in M,
3. $g^{-1}(f(F)) \cap \mathrm{int}(D) = \emptyset$.

LEMMA 4.6. *Up to homotopy there exists at most one elementary (resp. simple) compression of α.*

Proof. The results of Hass, Rubinstein and Scott [20] show that if X is the closure of a component of $M - f(F)$ then it is a handlebody and that $J = \partial X \cap f(S(f))$ is a finite graph of degree three on ∂X. Since $f(F)$ lifts to a simple family of planes satisfying the 1-line

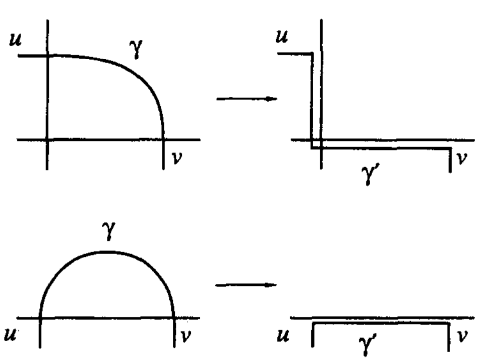

Fig. 10. Modifying γ by elementary and simple compressions.

property it is easy to see that J is non-degenerate in the sense of Section 3. The result then follows immediately from Theorem 3.4. Q.E.D.

Let γ be an embedded loop in M which is in general position with respect to $f(F)$ and α a segment of γ with endpoints x and y on $f(F)$. If α admits an elementary (resp. a simple) compression $g\colon D \to M$ then let γ' be the loop obtained by replacing α in γ by α' and moving the resulting loop slightly into general position with respect to $f(F)$ in such a way that $n(\gamma')$ is minimized. It is clear that γ and γ' are homotopic and that $n(\gamma) = n(\gamma')$ (resp. $n(\gamma') = n(\gamma) - 2$). We say that γ' is obtained from γ by an *elementary* (*resp. simple*) *compression* of α (Fig. 10). An arc α in M which lifts to an outermost arc in $\tilde{M}$ is said to be *outermost.*

LEMMA 4.7. *Let γ be a loop in M which is in general position with respect to $f(F)$ and α an outermost sub-arc of γ. If $\tilde{\alpha}$ is a lift of α to $\tilde{M}$ which is spanned by a disc $g\colon D \to \tilde{M}$ admitting a triangular compression of length $l = l(g)$ then it is possible to modify γ by a sequence of $l - 1$ elementary compressions followed by one simple compression to a loop γ' with $n(\gamma') = n(\gamma) - 2$.*

Proof. We must find a sequence of loops $\gamma_0 = \gamma, \gamma_1, \ldots, \gamma_{l-1}, \gamma_l = \gamma'$ such that,

1. γ_i is obtained from γ_{i-1} by an elementary compression for each $i = 1, \ldots, l - 1$,
2. γ_l is obtained from γ_{l-1} by a simple compression and
3. $n(\gamma') = n(\gamma) - 2$.

Let $P \in \Pi$ denote the plane on which the endpoints x and y of $\tilde{\alpha}$ lie and let $\tilde{\alpha}'$ be the arc $g|\partial D - \operatorname{int}(s)$ where $s \subset \partial D$ is the arc such that $g|s = \tilde{\alpha}$. The disc $g\colon D \to \tilde{M}$ admits a triangular compression,

$$g_0 = g\colon D_0 = D \to \tilde{M}, \ldots, g_l\colon D_l \to \tilde{M}$$

of length l. For each $i = 1, \ldots, l$ the disc g_i is obtained from g_{i-1} by an elementary

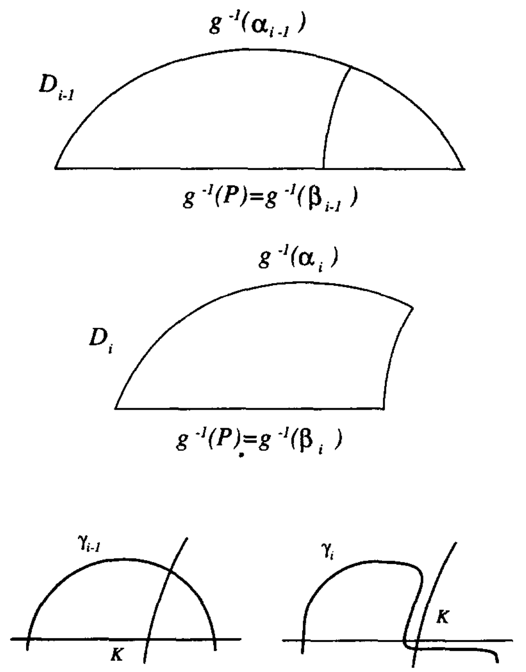

Fig. 11. Modifying $\tilde{\gamma}_{i-1}$ to obtain $\tilde{\gamma}_i$.

compression and spans an outermost arc $\tilde{\alpha}_i$ with endpoints x and y_i on P. Let $s_i \subset \partial D_i$ be the arc such that $g_i | s_i = \tilde{\alpha}_i$ and let $\tilde{\beta}_i = g_i | \partial D_i - \text{int}(s_i)$. Let $\tilde{\gamma}$ be a lift of γ to $\tilde{M}$ which contains $\tilde{\alpha}$ as a subarc and consider the sequence of arcs $\tilde{\gamma}_0 = \tilde{\gamma}, \ldots, \tilde{\gamma}_l$ where $\tilde{\gamma}_i$ is obtained by replacing the arc $\tilde{\alpha}_{i-1}$ in $\tilde{\gamma}_{i-1}$ by the arc $\tilde{\alpha}_i \cup (\tilde{\beta}_{i-1} - \tilde{\beta}_i)$ and moving the resulting arc slightly as shown in Fig. 11 so that it lies in general position with respect to K.

For each $i = 0, \ldots, l$ let γ_i denote the projection of $\tilde{\gamma}_i$ to a loop in M. Clearly for each $i = 1, \ldots, l-1$, γ_i is obtained from γ_{i-1} by an elementary compression and γ_l is obtained from γ_{l-1} by a simple compression. It follows that $\gamma_0, \ldots, \gamma_l$ is the required sequence. Q.E.D.

Lemma 4.8. *Let γ be an embedded loop in M which is in general position with respect to $f(F)$ and α a segment of γ with endpoints x and y on $f(F)$. Then it is possible to decide whether or not γ can be modified by an elementary (resp. simple) compression of α to a loop γ'. If it is possible then γ' can be constructed.*

Proof. Since $\text{int}(\alpha)$ is disjoint from $f(F)$, α is properly embedded in the closure X of a component of $M - f(F)$. The results of Hass, Rubinstein and Scott [20] show that X is a π_1-injective handlebody. The double curves of $f(F)$ describe a 3-valent finite graph $J = \partial X \cap f(S(f))$ on ∂X. We can apply Waldhausen's algorithm $\mathcal{U}_1(\alpha)$ in X [46] to decide whether or not α is homotopic to an arc α' from x to y on ∂X which crosses G at most once and if this is the case to construct α'. Furthermore we can ensure that if α' exists it can be constructed in such a way that it lifts to an embedding in $\tilde{M}$. It follows that α admits either a simple or an elementary compression and using α' we can modify γ as shown in Fig. 12 to give a loop γ' homotopic to γ and in general position with respect to $f(F)$. Q.E.D.

Lemma 4.9. *Assume that for each $k \geq 0$ there is a number $\lambda(k)$ such that any outermost arc α in M with $n(\alpha) = k$ is spanned by a disc g which admits a triangular compression of length $l(g) \leq \lambda(k)$. Then if γ is a loop in M which is in general position with respect to $f(F)$ it is possible to decide whether or not γ has a sub-arc which is outermost and if so to construct a loop γ' homotopic to γ and with $n(\gamma') = n(\gamma) - 2$.*

Proof. If $n(\gamma) = 2l$ and α is an outermost sub-arc of γ then $n(\alpha) \leq l - 1$. If α lifts to an outermost arc $\tilde{\alpha}$ in $\tilde{M}$, then it is spanned by a disc $g: D \to \tilde{M}$ admitting a triangular compression with length $l(g) \leq \lambda(n(\alpha)) \leq \lambda(l-1)$. By Lemma 4.7, γ can be modified by a sequence of $l(g) - 1$ elementary compressions followed by one simple compression to obtain a loop γ' homotopic with γ and with $n(\gamma') = n(\gamma) - 2$.

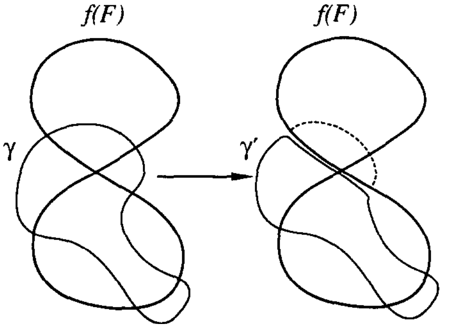

Fig. 12. Modifying γ to obtain γ'.

Using Lemma 4.8 we can decide whether or not γ can be modified by a sequence of at most $\lambda(l-1)-1$ elementary compressions followed by one simple compression and if so find the shortest such sequence. Note that by Lemma 4.6 an elementary compression of a segment of γ, if it exists, is unique so that there are only finitely many ways of choosing such a sequence. If there is no such sequence then γ can have no outermost arc. On the other hand, if there is such a sequence then clearly γ has an outermost arc and it is possible to modify γ using the sequence of compressions to give a loop γ' with $n(\gamma') = n(\gamma) - 2$. Q.E.D.

LEMMA 4.10. *It is possible to decide whether or not an embedded loop γ which is in general position with respect to $f(F)$ is contractible.*

Proof. We argue by induction on $n(\gamma)$.

If $n(\gamma) = 0$ then $\gamma \cap f(F) = \emptyset$ so that γ must be contained in the closure X of a component of $M - \mathcal{N}(f(F))$. Since the inclusion $X \subset M$ induces an injection $\pi_1(X) \to \pi_1(M)$ [20] it follows that γ is contractible in M if and only if it is contractible in X. But since X is Haken we can use Waldhausen's algorithm [46] to decide whether or not γ is contractible in X.

Suppose now that $n(\gamma) = k$. Any trivial loop in M lifts to a loop in $\tilde{M}$ which by Lemma 4.5 must have a sub-arc which is outermost. It follows that some sub-arc of a contractible loop must be outermost. Using Lemma 4.9 we can decide whether or not γ has a sub-arc which is outermost and if so modify γ to give a loop γ' homotopic to γ and with $n(\gamma') = n(\gamma) - 2$. By induction we can decide whether or not γ' is contractible. Since γ' is homotopic to γ we can therefore decide whether or not γ itself is contractible. On the other hand if γ has no outermost sub-arc then it is not contractible. Q.E.D.

The main result of this paper is an immediate consequence of this.

TEHEOREM 4.11. *Let M^3 be a closed, P^2-irreducible, non-Haken 3-manifold and $f\colon F \to M$ a general position immersion of a closed surface $F^2 \neq S^2, P^2$ into M such that the full pre-image of $f(F)$ in the universal cover $\tilde{M}$ of M is a simple family of planes $\Pi = \{P_i\}$ satisfying the 1-line and 1-point intersection properties. Then the Word Problem in M is solvable.*

5. EXTENSIONS

If extra assumptions are made on the surface $f\colon F \to M$ the argument of Section 4 can be improved. In this section we give two more versions of Lemma 4.4. The remainder of the algorithm to Theorem 4.11 follows as before.

5.1. The case where Π satisfies the 4-plane property

We assume now that Π satisfies the 4-plane and 1-line intersection properties. Since the techniques of Hass and Scott [21] show that Π can be placed in canonical position in which it also satisfies the 1-point property, the 4-plane property is actually a special case of the situation we have already considered so that the Word Problem is again solvable using Theorem 4.11. However in Lemma 4.4 only an exponential bound on the length of a triangular compression could be achieved. With the 4-plane property this can be improved considerably.

LEMMA 5.1. *Let α be an outermost arc with $n(\alpha) = k$ and endpoints x and y on a plane $P \in \Pi$. If $P_1, \ldots, P_k$ denote the k distinct planes from Π which* int(α) *intersects then α is*

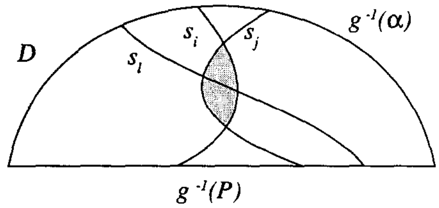

Fig. 13. 2-gon on D.

spanned by a disc $g\colon D \to \tilde{M}$ such that $\beta_1 = g^{-1}(P_1), \ldots, \beta_k = g^{-1}(P_k)$ are distinct properly embedded arcs on D any two of which intersect in at most one point.

Proof. The existence of a disc $g\colon D \to \tilde{M}$ spanning α follows from Lemma 4.4 since Π satisfies the 1-point property. Furthermore, it is clear that g can be chosen so that,

1. that no component of $g^{-1}(P_i)$, for any i, is a simple closed curve and,
2. if D' is the closure of a component of $D - g^{-1}(K)$ with $D' \cap \partial D = \emptyset$ then $g\,|\,D'$ is an embedding.

Suppose that for $1 \le i < j \le k$ the arcs β_i and β_j intersect in more than one point. Since β_i and β_j are properly embedded it is clear that there must be a 2-gon region D' on D bounded by arcs of β_i and β_j. If another of the arcs β_l with $(l \neq i, j)$ intersects D' in such a way that it meets both of the edges of D' then the 4-plane intersection property is violated since this would require the four planes P, P_i, P_j and P_l to intersect pairwise (Fig. 13). This means we can choose among all such 2-gons between two of the arcs $\beta_1, \ldots, \beta_k$ one (with no loss of generality we may assume that it is D') which is innermost (Fig. 13).

Both vertices of D' map under g to points on the line $P_i \cap P_j$ so that there is a region R of $\tilde{M}$ bounded by $g(D')$ and discs D_i and D_j on the planes P_i and P_j respectively. If another plane $Q \in \Pi$ intersects R then since $Q \cap g(D') = \emptyset$, $Q \cap R$ must consist of at least one properly embedded disc D'' whose boundary consists of an arc from D_i and an arc from D_j (Fig. 14). This disc D'' is thus a 2-gon region on the plane Q whose boundary consists of an arc from each of the lines $P_i \cap Q$ and $P_j \cap Q$. It follows that these lines must intersect in at least two points contradicting the assumption that Π satisfies the 1-point property.

By pushing $g(D)$ across R as shown in Fig. 15 we can alter g by a homotopy in such a way that the 2-gon D' is removed by pulling β_i and β_j apart, without otherwise changing the intersection pattern of the arcs $\beta_1, \ldots, \beta_k$. By repeating this operation at most finitely many times it is thus possible to remove all 2-gon regions on D and ensure that any two arcs from $\beta_1, \ldots, \beta_k$ intersect in at most one point. Q.E.D.

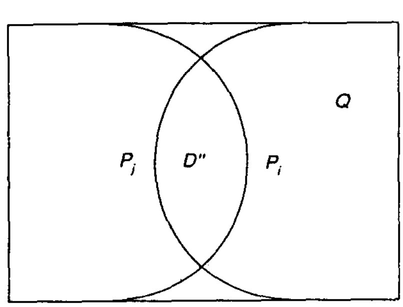

Fig. 14. The intersection of the region R with a plane $Q \in \Pi$.

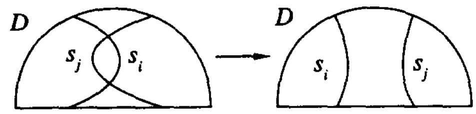

Fig. 15. Removing a 2-gon region on D.

LEMMA 5.2. *The disc g admits a triangular compression of length $l(g) \leq \frac{1}{2}k^2 + 1$.*

Proof. The existence of a triangular compression follows from the observation that if g has no upper triangle then it is a simple disc. To see this consider the configuration of the arcs $\beta_1, \ldots, \beta_k$ on D. Let β_1 be the last arc which meets $s = g^{-1}(\alpha')$. If β_1 does not intersect any of the other arcs $\beta_2, \ldots, \beta_k$ then the closure of the component of $D - g^{-1}(K)$ which is disjoint from $\beta_2, \ldots, \beta_k$ is an upper triangle of g (Fig. 16a). If β_1 does meet other arcs then these must be pairwise disjoint because of the 4-plane property. As shown in Fig. 16b there is again an upper triangle of g. To see that $l(g) \leq \frac{1}{2}k^2 + 1$ note that each elementary compression reduces by one either the number of double points between the arcs $\beta_1, \ldots, \beta_k$ or the number of points of intersection between the arcs $\beta_1, \ldots, \beta_k$ and s. But the total number of these points is at most $\frac{1}{2}k^2$. Q.E.D.

5.2. *The case where Π satisfies the 4-plane and triple-point properties*

We now assume that Π satisfies the triple-point property as well as the 4-plane and 1-line properties.

LEMMA 5.3. *Let α be an outermost arc with $n(\alpha) = k$ and endpoints x and y on a plane $P \in \Pi$. If $P_1, \ldots, P_k$ denote the k distinct planes from Π which* int(α) *intersects then α is spanned by a disc $g\colon D \to \tilde{M}$ such that $\beta_1 = g^{-1}(P_1), \ldots, \beta_k = g^{-1}(P_k)$ are disjoint properly embedded arcs on D.*

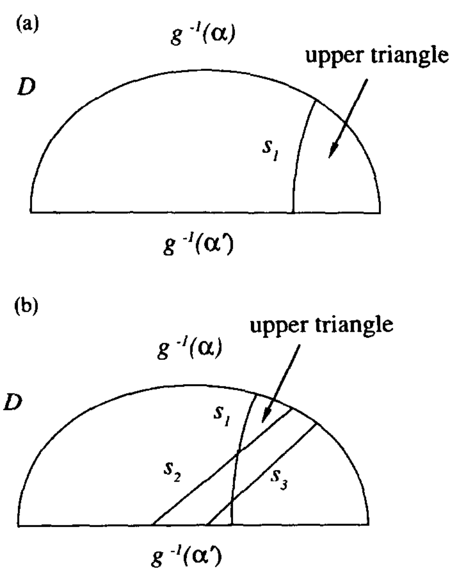

Fig. 16. Upper triangles of g.

Proof. We argue by induction on $n(\alpha)$.

The result is clearly true if $n(\alpha) = 0$.

Let $n(\alpha) > 0$ and assume that α is oriented in the direction from x to y and that it intersects the planes $P_1, \ldots, P_k$ in points $x_1 = P_1 \cap \alpha, \ldots, x_k = P_k \cap \alpha$ in that order from x to y.

Let $l = P \cap P_1$ and let γ be an embedded arc in P from x to l which is in general position with respect to Λ_P and whose interior meets L_P in a minimal number $n(\gamma)$ of points. Suppose that $n(\gamma) > 0$. If γ is oriented in the direction from x to l let l' be the first line from Λ_P which $\mathrm{int}(\gamma)$ meets and Q the plane from Π such that $Q \cap P = l'$. If $l \cap l' = \emptyset$ it follows from the triple-point property that $Q \cap P_1 = \emptyset$ since otherwise the three planes P, P_1 and Q would intersect pairwise without having a point in common. In this case l' cannot separate x from l since then Q would also separate x from x_1 and by Lemma 4.2, α would have to intersect Q between x and x_1. So since l' cannot separate x from l it is possible by Lemma 4.3 to find an arc from x to l whose interior does not intersect l' and hence meets L_P in fewer points than γ. This contradicts the assumption that $n(\gamma)$ was minimal and it follows that $l \cap l' \neq \emptyset$. Since Π satisfies the 1-point property, l and l' must intersect in a single point $z = P \cap P_1 \cap Q$. If z' is the first point of $\mathrm{int}(\gamma)$ which intersects l' then there must exist a line l'' from Λ_P which meets l' in a single point between z and z' since otherwise it would be possible as shown in Fig. 17a to choose γ with $n(\gamma) = 0$. As before it is not possible that $l'' \cap l = \emptyset$.

If R is the plane such that $R \cap P = l''$ we have four planes P, P_1, Q and R no two of which are disjoint (Fig. 17b). Since this contradicts the 4-plane property it follows that $n(\gamma) = 0$. Hence we can find an embedded arc α_1' from x to l whose interior does not intersect L_P.

Now suppose that x_1' is the endpoint of α_1' on l and that there exists a plane Q from Π which intersects P_1 in a line separating x_1' from x_1. This leads to a contradiction since Q must then intersect either α_1 or α_1'. Hence, by Lemma 4.3, there exists an embedded arc α_1'' from x_1 to x_1' on P_1 whose interior is disjoint from L_{P_1}.

Clearly there is a disc $g_1\colon D_1 \to \tilde{M}$ with $g_1 | \partial D_1 \alpha_1 \cup \alpha_1' \cup \alpha_1''$ as shown in Fig. 18a. Consider the arc β obtained by replacing the sub-arc α_1 of α by the arc $\alpha_1' \cup \alpha_1''$. If we move the sub-arc α_1'' of β slightly off P_1 as shown in Fig. 18b we see that β consists of an arc β_1 from x to y_1 on P with $n(\beta_1) = 1$ and an arc β_2 from y_1 to y which is in general position

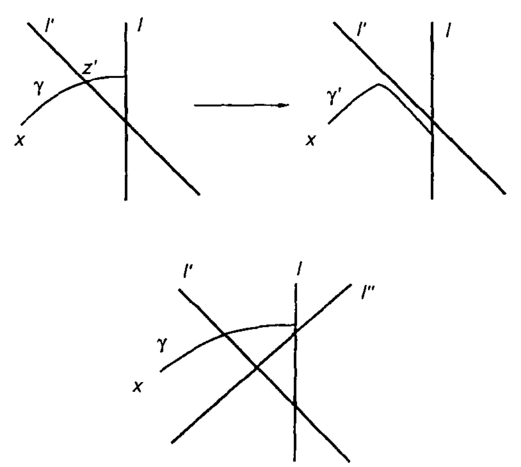

Fig. 17. The intersection of γ with Λ_P.

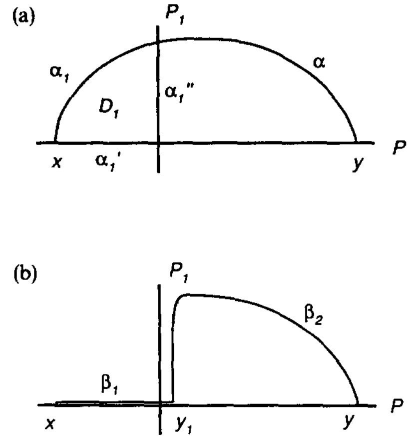

Fig. 18. Modifying α.

with respect to Π and with $n(\beta_2) = k - 1$. Clearly β_2 is an outermost arc which meets the planes $P_1, \ldots, P_k$ in the points $x_2, \ldots, x_k$ in order from y_2 to y. By induction β_2 is spanned by a disc $g_2 : D_2 \to \tilde{M}$ such that the arcs $\beta_2 = g_2^{-1}(P_2), \ldots, \beta_k = g_2^{-2}(P_2)$ are all disjoint.

The result follows if we let $g : D \to \tilde{M}$ be the disc obtained by pasting g_1 and g_2 along the arc α_1'' in each of their boundaries. Clearly g spans α and the arcs $\beta_1 = g^{-1}(P_1), \ldots, \beta_k = g^{-1}(P_k)$ are all disjoint. Q.E.D.

LEMMA 5.4. *The disc g above admits a triangular compression of length exactly $l(g) = k + 1$.*

Proof. This is clear since the disc D_1 is an upper triangle for g. Q.E.D.

6. THE SPEED OF THE ALGORITHM

The number of steps which are needed for the algorithm in Theorem 4.11 to decide whether or not a loop in M is contractible depends essentially on the bounds on $l(g)$ given in Lemma 4.4, Lemma 5.2 and Lemma 5.4. We will write $l(k)$ here for $l(g)$ to emphasise the dependence on k.

An effective way of measuring the speed of the algorithm is to find an upper bound $b(k)$, over all loops γ with $n(\gamma) = 2k$, on the minimum number of cells in the cellulations $g^{-1}(f(F))$ given by singular discs $g : D \to M$ bounded by γ. If $n(\gamma) = 2k$ and γ is contractible, then by Lemma 4.5 there exists an outermost arc α of γ with $n(\alpha) \leq k - 1$. Lemma 4.7 allows us to modify γ to give a loop γ' with $n(\gamma') = 2k - 2$. Since this is achieved by a triangular compression of length at most $l(k - 1)$ it follows that we can choose a disc $g : D \to M$ bounded by γ so that the number of cells in the cellulation $g^{-1}(f(F))$ is at most $b(k - 1) + l(k - 1)$. Clearly then $b(k) = b(k - 1) + l(k - 1)$, and since $b(0) = 1$, we have that,

$$b(k) = 1 + \sum_{j=0}^{k-1} l(j).$$

Using the bound $l(j) \leq 2^j$ of Lemma 4.4 we can calculate that $b(k) = 2^k$. A similar calculation, using the bound on $l(g)$ given in Lemma 5.2 and 5.4, gives $b(k) = \frac{1}{6}k(k - 1)(2k - 1)$ and $b(k) = \frac{1}{2}(k^2 + k + 2)$, respectively.

It seems likely that these values for $b(k)$ will correspond respectively to exponential, cubic and quadratic isoperimetric inequalities.

7. EXAMPLES

In this section we apply Theorem 4.11 to solve the Word Problem in a class of closed 3-manifolds which were not previously known to have solvable Word Problems.

A typical example of the class of manifolds considered in Section 4 is a non-Haken hyperbolic manifold M which contains an immersed totally geodesic surface. Certainly this surface will satisfy the 1-line and 1-point intersection properties [21]. However in this case the Word Problem is solvable by other means. Using the hyperbolic structure in M, any loop in M can be algorithmically shortened until it has length which is arbitrarily close to the length of the unique closed geodesic in its homotopy class. A loop is then seen to be trivial if and only if it can be shortened to have length smaller than the length of any non-trivial closed geodesic.

Less well known examples, due to Aitchison and Rubinstein are provided by manifolds which admit a cubing of non-positive curvature [1], [3] and [4]. Such a manifold has a decomposition into Euclidean cubes, with face identifications given by Euclidean isometries, in such a way that,

1. each edge must belong to at least four cubes and,
2. the polyhedral link $lk(v)$ of each vertex contains no cycles of length less than four unless they are boundaries of triangular faces of $lk(v)$.

Each such manifold M contains an immersed, π_1-injective surface $f\colon F \to M$ satisfying the 1-line, 4-plane and triple-point properties. The surface is constructed by taking three squares parallel to and midway between the opposite faces of each cube and identifying the squares in adjacent cubes using the face identification isometries (Fig. 19). The resulting two complex is the image $f(F)$ of f in M and is seen to be π_1-injective, since it is totally geodesic in the singular metric of M [3]. In this case the Word Problem is again solvable without using Theorem 4.11 since it is possible to constructively build arbitrarily large pieces of the universal covers of M. Given a loop in M we test to see whether or not it is contractible by constructing its lift to $\tilde{M}$ and checking whether this is again a loop.

Non-trivial examples can however, be obtained by taking semi-ideal cubings of certain link complements [1]. More precisely we take a 3-valent polyhedron P which has no facets with fewer than four edges. Furthermore we require that the edge graph, G, of P is

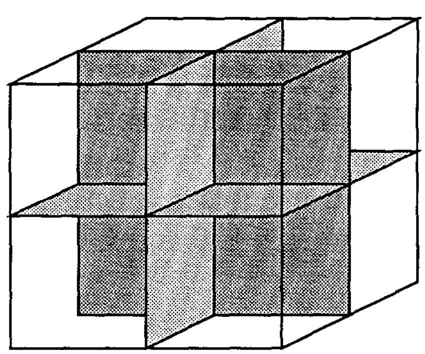

Fig. 19. The intersection of $f(F)$ with a cube

non-degenerate, in the sense that any disc on ∂P whose boundary meets G in fewer than four points intersects G is one of three ways shown in Fig. 20.

The facets of P are provided with a 2-colouring (black/white) in such a way that no vertex is surrounded by facets all of the same colour (Fig. 21).

Two copies of P_1 and P_2 of P, with vertices deleted, are then identified along their boundaries in such a way that if an n-gon facet of P_1 coloured black (resp. white) then it is identified to the corresponding face in P_2 after a clockwise rotation of $2\pi/n$ (resp. $-2\pi/n$). The resulting space is homeomorphic to the complement of an alternating link L in S^3. The link, L, is obtained from the edge graph, G, of P by replacing each edge of G which is adjacent to two black (resp. white) faces by a "clasp" as shown in Fig. 22.

A cubical decomposition of P is obtained by taking the graph, G', dual to the edge graph of P and coning this to a point x in the center of P. Each cell in the resulting decomposition of P is a tetrahedron with one face in ∂P. However, the graph $G \cup G'$ on ∂P subdivides this face into three squares and it is clear (Fig. 23) that each cell can be considered a cube.

Since P_1 and P_2 are copies of P, this gives a decomposition of $S^3 - L$ into a finite collection of cubes, each with one deleted vertex. A (singular) metric can be placed on this space by requiring that each cube is isometric to a semi-ideal cube (with one vertex at infinity) in hyperbolic three-space (Fig. 24) and that the identifications between adjacent cubes are isometries. By using a similar construction to that used for standard cubings, $S^3 - L$ is seen to contain an immersed, π_1-injective surface f: $F \to M$ satisfying the 1-line, 4-plane and triple-point properties. Moreover, it is clear that $f(F)$ must be a closed surface.

Let $\mathcal{N}$ denote a regular neighbourhood of $f(F)$ in $S^3 - L$ and M the 3-manifold obtained by removing from $S^3 - L$ those components of $(S^3 - L) - \mathcal{N}$ containing cusps. It is clear that M is homeomorphic to $S^3 - L$ and that the boundary, ∂M, of M is a union of tori. Consider the projection of $S(f)$ onto the boundary of $\mathcal{N}$. This describes a hexagonal tiling of each component torus of ∂N as shown in Fig, 25. Let N denote a closed 3-manifold obtained by performing Dehn surgeries on each component of ∂M in such a way that each surgery curve can not be homotoped to a curve meeting the edges of this tiling in fewer than four points. Then by an observation of Aitchison, Rubinstein and Thurston [6], the map f: $F \to N$ is π_1-injective and retains the 4-plane, 1-line and triple point properties. It follows that if N is non-Haken, the word problem in N is solvable by Theorem 4.11. Note that only finitely many surgeries on each boundary component of M are disallowed. In general it is

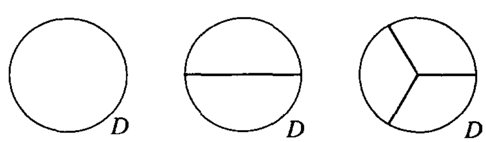

Fig. 20. Allowable intersection of a disc with G.

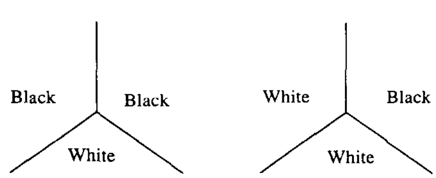

Fig. 21. Allowable colourings around a vertex.

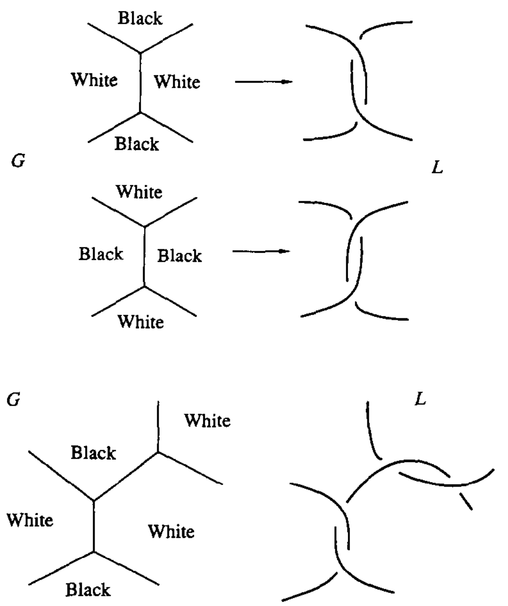

Fig. 22. Obtaining L from the edge graph on P.

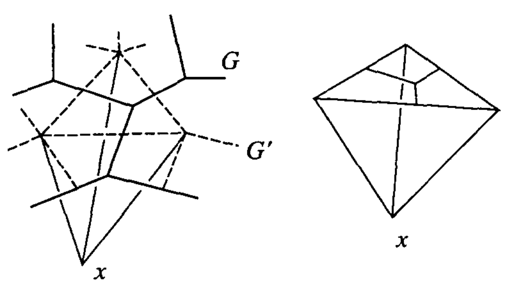

Fig. 23. Decomposition of $S^3 - L$ into cubes.

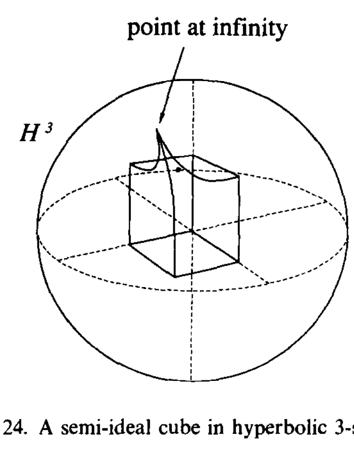

Fig. 24. A semi-ideal cube in hyperbolic 3-space.

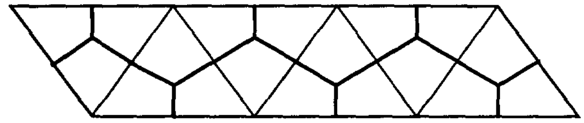

Fig. 25. The hexagonal tiling of a boundary torus.

not known whether or not a 3-manifold N obtained in this way admits a metric of non-positive curvature.

This procedure is a combinatorial version of Thurston's Hyperbolic Dehn surgery (the Gromov–Thurston 2π-theorem [7]) which ensures that if sufficiently long surgery curves on each boundary component of M are chosen, the resulting surgered manifold, N, has non-positive curvature. In this case, although the surface f again persists, Theorem 4.11 is unnecessary since the Word Problem in N is solvable due to the non-positively curved geometry. Again for each boundary component of M there are only finitely many surgeries which are excluded. However, in some cases there are a small (finite) number of surgeries on a boundary component which are excluded by Hyperbolic Dehn surgery but not by "combinatorial" Dehn surgery. As a result there are infinitely many examples of closed 3-manifolds which contain singular, incompressible surfaces satisfying the 4-plane, 1-line and triple-point properties and which are not known to have metrics of non-positive curvature. These give a class of non-trivial examples to which Theorem 4.11 can be applied.

A more general class of examples can be obtained by another observation of Aitchison, Rubinstein and Thurston [6]. Let $f\colon F \to M$ be a general position immersion of a closed surface, F, into a 3-manifold, so that $F - S(f)$ has no simply-connected regions with fewer than four edges. Assume that any embedded disc D in M, with $D \cap f(F) = \partial D$, which intersects $f(S(f))$ in fewer than four points is homotopic to a disc in $f(F)$ which intersects $f(S(f))$ in one of three patterns shown in Fig. 24. Then f is π_1-injective and satisfies the 4-plane, 1-line and triple-point properties. We can use this result in the following way. If $f\colon F \to M$ is a general position immersion for which no simply connected region of $F - S(f)$ has fewer than four edges, let $\mathcal{N}$ denote the closure of a regular neighbourhood of $f(F)$. The boundary of $\mathcal{N}$ consists of finitely many closed surfaces on each of which is inscribed a 3-valent graph obtained by projecting the image of $S(f)$ onto $\partial\mathcal{N}$. Then if a closed 3-manifold, M, is obtained by glueing handlebodies onto the boundary components of $\mathcal{N}$ in such a way that each meridian disc meets the graph on $\partial\mathcal{N}$ essentially in at least four points, the resulting immersion $f\colon F \to M$ is π_1-injective and satisfies the 4-plane, 1-line and triple-point properties. In these cases there is no general method of providing M with a geometric structure, whether smooth or singular, However, the Word Problem is these examples can be solved again using Theorem 4.11.

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30 *Larnook Street*
Prahan 3181
Victoria
Australia